

# Alternative measurements of the fermion $g$ -factor in the field of a traveling circularly polarized electromagnetic wave

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The field of a traveling circularly polarized electromagnetic wave and a constant magnetic field localizes fermions perpendicularly to propagation of the wave in the cross section of the order of the wavelength. Unusual exact solutions of the Dirac equation correspond to this localization. Except to routine use of thin fermion beams it can be suitable for alternative measurements of the  $g$  - factor. Details and peculiarities of the solutions in application to the measurements are considered in the paper.

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## INTRODUCTION

Measurements of the magnetic moment began from works of I. Rabi, who developed the resonance radio-frequency techniques before World War II. The technique has been significantly refined with the advent of quantum electrodynamics [1].

The use of a rotating coordinate system to solve magnetic resonance problems is described in [2]. The time-dependent eigenvalue problem may turn into a stationary with help of such a system.

This approach makes it possible to find an exact localized solution of Dirac equation in the field of a traveling circularly polarized electromagnetic wave and constant magnetic field [3].

Temporal behavior of spin in such an electromagnetic field differs principally from that in the standard magnetic resonance with rotating and constant magnetic field described by the Pauli equation. This opens the way for an alternative measurement of the fermion  $g$ -factor.

The present paper summarizes the peculiarities of the solutions applicable for such measurements.

### The Dirac equation and potential

Consider Dirac's equation

$$\{-i\frac{\partial}{\partial t} - i\boldsymbol{\alpha}\frac{\partial}{\partial \mathbf{x}} - \boldsymbol{\alpha}\mathbf{A} + \beta\}\Psi = 0 \quad (1)$$

in the electromagnetic field with the potential

$$A_x = -\frac{1}{2}H_z y + \frac{1}{\Omega}H \cos(\Omega t - \Omega z), \quad (2)$$

$$A_y = \frac{1}{2}H_z x + \frac{1}{\Omega}H \sin(\Omega t - \Omega z). \quad (3)$$

$H_z$  is the constant component of the magnetic field along the  $z$ -axis,  $H$  is the amplitude of the circularly polarized component.

We use normalized dimensionless variables and parameters:  $(ct, x, y, z) \rightarrow (ct, x, y, z)/\lambda$ ,

$$\mathbf{A} \rightarrow \frac{e\lambda}{c\hbar}\mathbf{A}, \quad \frac{e\lambda^2}{c\hbar}\mathbf{H} \rightarrow \mathbf{H}, \quad \Omega \rightarrow \Omega\frac{\lambda}{c} = 2\pi\frac{\lambda}{\lambda}, \quad (4)$$

$$E \rightarrow \frac{1}{mc^2}E, \quad p \rightarrow \frac{1}{mc}p, \quad d \rightarrow \lambda^2 d, \quad d_2 \rightarrow \lambda d_2, \quad (5)$$

$\lambda$  is the Compton wavelength  $\lambda = \hbar/mc$ ,  $\lambda$  is the wavelength of the electromagnetic wave,  $c$  is the speed of light,  $\hbar$  is the reduced Planck constant. The charge  $e$ , for definiteness, is assumed to be negative  $e = -|e|$ . In the dimensionless units the propagation constant equals  $\Omega$ .

The potential  $\mathbf{A}$  describes a traveling circularly polarized electromagnetic wave propagating along the constant magnetic field  $H_z$ .

## SOLUTIONS

The Dirac's equation (1) has exact solutions localized in the cross section perpendicular to the propagation direction of the wave [3]. In the lab frame only non-stationary states are possible. In contrast to that in the rotating frame stationary states exist. The solutions in the lab frame can be presented as follows

$$\Psi = \exp[-iEt + ipz - \frac{1}{2}\alpha_1\alpha_2(\Omega t - \Omega z) + D]\psi, \quad (6)$$

where constants  $E$  and  $p$  are "energy" and "momentum" in the rotating coordinate system,

$$D = -\frac{d}{2}r^2 - id_2\tilde{x} + d_2\tilde{y}, \quad (7)$$

parameters  $d, d_2$  are determined by substitution the wave function (6) in Eq. (1)

$$d = -\frac{1}{2}H_z, \quad d_2 = \frac{\mathcal{E}_0 h}{2(\mathcal{E} - \mathcal{E}_0)}, \quad h = \frac{1}{\Omega}H, \quad (8)$$

$$\tilde{x} = r \cos \tilde{\varphi}, \quad \tilde{y} = r \sin \tilde{\varphi}, \quad \tilde{\varphi} = \varphi - \Omega t + \Omega z. \quad (9)$$

The constant spinor  $\psi$  described the "ground state", spinor polynomials in  $\tilde{x}$  and  $\tilde{y}$  correspond to the "excited states".

Below only the ground state is considered. In this case the wave function has the form

$$\psi = N \begin{pmatrix} h\mathcal{E} \\ -(\mathcal{E} + 1)(\mathcal{E} - \mathcal{E}_0) \\ h\mathcal{E} \\ -(\mathcal{E} - 1)(\mathcal{E} - \mathcal{E}_0) \end{pmatrix}, \quad (10)$$

$N$  is a normalization constant, determined by the normalization integral  $\int \Psi^* \Psi ds = 1$ ,

$$N^2 [h^2 \mathcal{E}^2 + (\mathcal{E}^2 + 1)(\mathcal{E} - \mathcal{E}_0)^2] \frac{\pi}{d} \exp\left(\frac{d^2}{d}\right) = 1, \quad (11)$$

$E$  obeys the characteristic equation

$$\mathcal{E}(\mathcal{E} + 2p - \Omega) - 1 - \frac{\mathcal{E}h^2}{\mathcal{E} - \mathcal{E}_0} = 0, \quad (12)$$

$$\mathcal{E} = E - cp, \quad \mathcal{E}_0 = \frac{2d}{\Omega}. \quad (13)$$

It is noteworthy that Eq. (12) is algebraic equation of the third order, in contrast to the even order in many other cases.

#### $g$ - factor

The parameter  $\mathcal{E}_0$  in the non-normalized units is defined as

$$\mathcal{E}_0 = \frac{2\mu H_z}{\hbar\Omega}, \quad (14)$$

where  $\mu = |e|\hbar/(2mc)$  is the Bohr magneton. The definition (14) by equating  $2/\mathcal{E}_0$  to the  $g$  - factor turns into the classical condition of the magnetic resonance

$$\hbar\Omega = g\mu H_z. \quad (15)$$

This condition follows from the Pauli equation. This equation is the nonrelativistic approximation of the Dirac equation at  $g/2 = 1$ . As it is known from experiments  $g/2$  differs from 1. This fact has been explicitly explained by quantum field theory.

In the experiment  $g$  - factor is calculated by means of values  $H_z$  and  $\Omega$  at a maximum amplitude of the spin flip-flop in the magnetic resonance.

Phenomenologically this result is inserted in the Pauli equation by multiplication of the magnetic energy  $\mu\sigma\mathbf{H}$  by the experimental value of the  $g$  - factor.

Unique feature of solution (10) is that this solution never can be presented as large and small two-component spinor, i. e., always corresponds to a relativistic case. An equivalent of inserting  $g$  - factor in the Pauli equation is the equating  $2/\mathcal{E}_0$  to the  $g$  - factor in the solution (10) of the Dirac equation.

#### Singular solutions

Evaluate the normalized parameter  $h$ . Typically, the amplitude of the magnetic field  $H$  is much smaller than the constant magnetic field  $|H_z|$

$$h = \frac{1}{\Omega} H \ll \frac{1}{\Omega} |H_z| = \mathcal{E}_0 \sim \frac{2}{g}, \quad (16)$$

If  $g \sim 2$  then  $h \ll 1$ . Therefore  $h$  is very small.

Typically  $\mathcal{E}$  is expanded in power series in  $h^2$ . However, there exists a pair solutions for which  $\mathcal{E}$  is expanded in power series in  $h$

$$\mathcal{E} = \mathcal{E}_0 + h\mathcal{E}_1 + h^2\mathcal{E}_2 + \dots, \quad \mathcal{E}_1 = \pm \frac{\mathcal{E}_0}{\sqrt{\mathcal{E}_0^2 + 1}}. \quad (17)$$

The solutions are singular solutions. Two solutions in the pair differ by the sign of  $\mathcal{E}_k$  with odd numbers.

The singular solutions are best suited to experiment.

For such a pair in the first approximation

$$d_2 \approx \pm \frac{\sqrt{\mathcal{E}_0^2 + 1}}{2}. \quad (18)$$

The necessary condition for existence of the expansion (17) is following expression for the parameter  $p$  (momentum in the rotating frame) for both the states in the pair.

$$p = \frac{1}{2} \left( \frac{1}{\mathcal{E}_0} - \mathcal{E}_0 \right) + \frac{\Omega}{2}. \quad (19)$$

With this momentum  $p$  the energy  $E$  also coincide but with accuracy  $\sim h$

$$E = \mathcal{E}_0 + p = \frac{1}{2} \left( \frac{1}{\mathcal{E}_0} + \mathcal{E}_0 \right) + \frac{1}{2} \Omega + \dots \quad (20)$$

$\Omega \ll \mathcal{E}_0$  (in the non-normalized units this inequality is  $\hbar\Omega \ll mc^2\mathcal{E}_0$ ) and the last term in (19) and (20) can be neglected.

In particular, the energy and momentum of electron in rotating frame in non-normalized units is  $E \approx mc^2$  and  $p \approx -1.165 \cdot 10^{-3} mc$ , because  $g/2$  differs from one by  $1.165 \cdot 10^{-3}$ .

Note that similar pairs also possible for the Pauli equation in rotating magnetic field. Sum and difference of wave functions of the pair produce states with the spin oscillations. Such states take part in the magnetic resonance [3].

However, in the given case the states have a vanishingly small amplitude. This amplitude is proportional the factor  $\exp(-2d_2^2/d)$ . In the non-normalized units

$$\frac{2d_2^2}{d} \rightarrow \frac{\mathcal{E}_0^2 + 1}{\mathcal{E}_0} \frac{\lambda}{2\pi\lambda}. \quad (21)$$

The typical frequency of the magnetic resonance  $\sim 100GHz$ , the wavelength corresponding to the frequency  $\lambda \sim 0.3 \text{ cm}$ . The ratio  $\lambda$  to Compton wavelength is of the order of  $10^9$ . Therefore the term  $\exp(-2d_2^2/d)$  is extremely small and this case is not considered in the paper.

## SPATIAL AVERAGING

For non-stationary solutions average values of operators should be used:  $\overline{P} = \int \Psi^* P \Psi dx dy$ . Here and below all integrations in respect to  $x$  and  $y$  are from  $-\infty$  to  $+\infty$  and small terms of the order of  $h$  and of the order of  $\hbar\Omega \ll mc^2$  are neglected. Results are presented in non-normalized units.

### Localization

The wave function (10) is one more example of "optimal localization" like the ground state of the harmonic oscillator. It means the equality in the uncertainty relation

$$\overline{(\Delta x)^2} \overline{(\Delta p_x)^2} = \frac{\hbar}{2}, \quad \overline{(\Delta y)^2} \overline{(\Delta p_y)^2} = \frac{\hbar}{2}. \quad (22)$$

The parameter of the localization may be determined as follows

$$2\sqrt{\int \Psi^*(x^2 + y^2) \Psi dx dy} = \frac{\lambda}{\pi} \sqrt{1 + \left(\frac{g}{2}\right)^2}. \quad (23)$$

For electron  $g \sim 2$  and this parameter equals  $\lambda \cdot 0.45016$

### Energy and momentum

Neglecting terms with  $h \ll 1$  and  $\hbar\Omega \ll mc^2$  obtain energy

$$\overline{E} \approx mc^2 \left( \frac{g}{2} + \frac{2}{g} \right) \quad (24)$$

and components of momentum

$$\overline{p_x} \approx \mp mc \frac{\sqrt{4+g^2}}{2g} \cos(\Omega t - \Omega z), \quad (25)$$

$$\overline{p_y} \approx \mp mc \frac{\sqrt{4+g^2}}{2g} \sin(\Omega t - \Omega z), \quad (26)$$

$$\overline{p_z} \approx mc \frac{g}{2}. \quad (27)$$

For electron  $\overline{E} \approx 2mc^2$ ,  $\overline{p_z} \approx mc$ .

### Spin

The average value of spin components are  $s_n = -\frac{i}{2}\hbar \int \Psi^* \sigma_n \Psi ds$ , where

$$\sigma_1 = \alpha_2 \alpha_3, \quad \sigma_2 = \alpha_3 \alpha_1, \quad \sigma_3 = \alpha_1 \alpha_2. \quad (28)$$

In the same conditions  $h \ll 1$  and  $\hbar\Omega \ll mc^2$  obtain

$$s_1 \approx \mp \frac{\hbar g}{2\sqrt{4+g^2}} \cos(\Omega t - \Omega z), \quad (29)$$

$$s_2 \approx \mp \frac{\hbar g}{2\sqrt{4+g^2}} \sin(\Omega t - \Omega z) \quad (30)$$

$$s_3 \approx 0. \quad (31)$$

It is noteworthy that

$$\frac{s_1}{\overline{p_x}} = \frac{s_2}{\overline{p_y}} = \lambda \frac{g^2}{(4+g^2)}. \quad (32)$$

For electron the right part of this equation  $\approx \lambda/2$ .

### Conclusion

With help of the traveling circularly polarized electromagnetic wave and constant magnetic field fermion can be localized in the cross section of the order of the wavelength. Fermion states in the are described by the singular solutions of the Dirac equation and must have a certain energy and momentum, moreover the classical condition of the magnetic resonance should be fulfilled.

Measurements of the  $g$ -factor are possible by varying the ratio  $H_z$  to  $\Omega$ . This results to the change of the spin amplitude as well as energy and momentum at output of the electromagnetic wave area.

Undoubtedly, accuracy of such measurements is less than that in traditional measurements, however, such measurements are of importance because it can give an answer on the question: is the anomalous magnetic moment a intrinsic characteristic of fermion or it is a subject of the interpretation of measurements?

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